New Framework for Modulated Perfect Reconstruction Filter Banks

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***Abstract*— A new formulation for the analysis and design of modulated filter banks is introduced in this paper. The formulation provides a broad range of design flexibility within a compact framework and allows for the design of a variety of computationally efficient modulated filter banks with different numbers of bands and virtually arbitrary lengths.**

A unique feature of the formulation is that it provides explicit control of the input-to-output system delay in conjunction with perfect reconstruction. Design examples are given to illustrate the methodology.

I. Introduction

THE concept of modulated filter banks has a long history, dating back to early work with transmultiplexers [1]. With the introduction of subband coding in 1976 [2], a new application for filter banks was born—one that is now a dominant driving force behind filter bank research in the signal processing community. Analysis/synthesis filter banks for subband coding are now very well known. In the analysis section of the subband coder, the input $x(n)$ is filtered by a set of analysis filters, $h_k(n)$ and then decimated by N to form the analysis section outputs $y_k(n)$. For convenience in our presentation, we define $h_k(n)$ to be time-reversed versions of the analysis filter impulse responses, i.e., $h_k(n) \longrightarrow h_k(-n)$. This leads to the analysis section outputs

$$y_k(n) = \sum_{m=-\infty}^{\infty} x(m) h_k(m - nN) \tag{1}$$

instead of the usual $\sum_{m=-\infty}^{\infty} x(m)h_k(nN - m)$. The coefficients h_k shown above are "time" reversed versions of the impulse responses because the filters are viewed as vectors here, which is useful later. The synthesis section of the subband coder performs the dual operations of upsampling, filtering, and merging. If $g_k(n)$ is the set of synthesis filters, the overall analysis/synthesis system has output

$$z(n) = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h_k(m - Nl)x(m) \cdot g_k(n - lN).$$

If the analysis/synthesis system is exactly reconstructing, then $z(n) = x(n - n_0)$, where n_0 is the system delay. There are several important issues to consider that impact the performance and cost effectiveness of subband analysis/synthesis filter banks in practical applications, such as audio compression, speech coding, and subband image coding. First, the quality of the individual filters in both banks should be good. Typically, this means having high stopband attenuation, good transition band properties, and/or good impulse response characteristics. Second, the overall analysis/synthesis system should reconstruct the input with negligible distortion in the absence of subband quantization. Third, the filter banks should have an efficient implementation. This impacts the speed of the system as well as the cost. Finally, the overall system delay should be considered. In speech coding applications, for example, perceptually noticeable delay can be a significant form of quality degradation.

It is difficult to properly balance all of these issues in the design of analysis/synthesis filter banks. There have been many contributions in the literature focusing on various aspects of the design problem [7], [8], [12]–[14], [17]–[20]. A variety of methodologies based on time domain representations, as well as frequency and z-domain formulations, are now available [3], [4], [8], [16], but all of these methods have limitations.

Constraining the filter banks to be modulated filters is an effective way of building in implementation efficiency into the design method. Modulated filter banks rely on a baseband filter that is implicitly modulated, via a transform (such as a DCT), to create the filter bank. They are typically very efficient computationally because fast transform algorithms are generally employed. A modulated analysis filter $h_k(n)$ and synthesis filter $g_k(n)$ have the general form $h_k(n) = h(n) \cdot \Phi_k(n)$ and $g_k(n) = h'(n) \cdot \hat{\Phi}_k(n)$, respectively, where $\Phi_k(n)$ and $\hat{\Phi}_k(n)$ are the modulation functions or kernels. The analysis and synthesis baseband filters $h(n)$ and $h'(n)$ are typically lowpass filters that are modulated to form bandpass filters. Modulation functions are typically cosines, sines, and exponentials and thus lead to very computationally efficient systems.

The earliest work in modulated analysis/synthesis filter banks reported in the conference literature was by Rothweiler [5]. Other work based on using the modulation concept followed, such as the TDAC filter bank of Princen *et al.* [7], the lapped transforms by Malvar [13], and many others [6], [12], [14], [16]–[20]. In *these* papers, issues related to analysis and design are addressed. In *this* paper, a new formulation is

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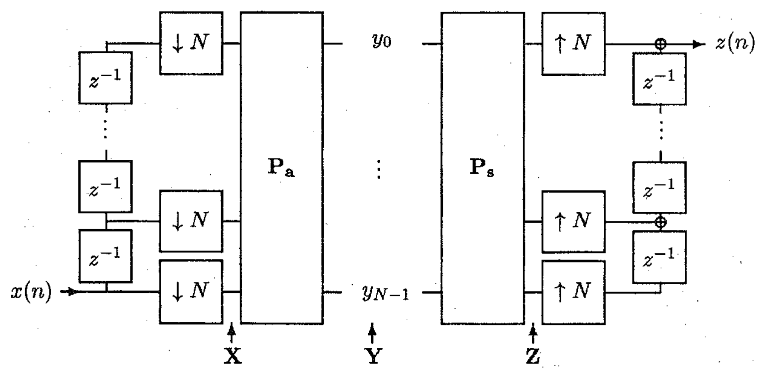

Fig. 1. Equivalent formulation of a classical filter bank.

introduced that attempts to provide a broader range of design flexibility than reported previously while maintaining a simple and compact framework. The new formulation allows for the design of a variety of efficient modulated filter banks with different numbers of bands and virtually arbitrary lengths. It also allows for the simultaneous control over the overall system delay for a given filter length. Such capability has not been present in other work with the exception of that reported in [12] by Nayebi *et al.* The approach developed here has some advantages over the method discussed in [12] in terms of filter quality, reconstruction quality, and implementation structure. Furthermore, it provides the first demonstration that low delay modulated filter banks can be designed with reconstruction that is exact in the mathematical sense.

II. The Basic Matrix Framework

The matrix framework discussed next is introduced to allow the various filter and system design objectives to be handled in a compact mathematical formulation. For an N-band analysis/synthesis filter bank, the input is represented by an N-dimensional row vector $\mathbf{x}(n)$ composed of the downsampled input components $\mathbf{x}(n) = [x(nN), x(nN+1), \cdots, x(nN+N-1)]$, where n may be viewed as the index of the downsampled sequences $x(nN+m)$, $m = 0, 1, \cdots, N-1$. This is illustrated graphically in Fig. 1, where $\mathbf{x}(n)$ is the vector input to the block denoted $\mathbf{P_a}$. Taking the z-transform of each element, we obtain the vector $\mathbf{X} = [X_0(z), \cdots, X_{N-1}(z)]$. For every block of N input samples, N output samples are produced. As shown in Fig. 1, these outputs are $y_k(n)$, where $k = 0, 1, \cdots, N-1$ are expressed as the vector $\mathbf{y}(n) = [y_0(n), \cdots, y_{N-1}(n)]$ with corresponding z-transform $\mathbf{Y} = [Y_0(z), \cdots, Y_{N-1}(z)]$.

The analysis filters $h_k(n)$ that convert $\mathbf{x}(n)$ into $\mathbf{y}(n)$ are contained in the analysis polyphase filter matrix $\mathbf{P_a}$. These filters are represented explicitly as having a length that is an integer multiple of the block length N. In particular, the length is represented by LN, where L is a positive integer. Filters with arbitrary lengths can be accommodated implicitly in the formulation by restricting an appropriate number of coefficients at the end to be zero. This point is clearly illustrated in the examples at the end of the paper. The specific form of the analysis polyphase filter matrix is (see also [8])

$$\mathbf{P_a} = \begin{bmatrix} P_{0,0}(z) & P_{0,1}(z) & \cdots & P_{0,N-1}(z) \\ P_{1,0}(z) & P_{1,1}(z) & & \\ \vdots & & \ddots & \\ P_{N-1,0}(z) & & \cdots & P_{N-1,N-1}(z) \end{bmatrix} \quad (2)$$

where the matrix elements are z-domain polynomials given by

$$P_{n,k}(z) = \sum_{m=0}^{L-1} h_k(n+mN) z^{-(L-1-m)}. \quad (3)$$

Given the input and output and polyphase filter matrix as defined, the analysis filter bank can be written as $\mathbf{Y} = \mathbf{X} \cdot \mathbf{P_a}$.

Similarly for the synthesis filter bank, the input is the vector $\mathbf{y}(n)$, and the synthesis filters $g_k(n)$ are contained in the synthesis polyphase filter matrix

$$\mathbf{P_s} = \begin{bmatrix} P'_{0,0}(z) & P'_{0,1}(z) & \cdots & P'_{0,N-1}(z) \\ P'_{1,0}(z) & P'_{1,1}(z) & & \\ \vdots & & \ddots & \\ P'_{N-1,0}(z) & & \cdots & P'_{N-1,N-1}(z) \end{bmatrix} \quad (4)$$

where $P'_{k,n}(z)$ are polynomials in z defined by

$$P'_{k,n}(z) = \sum_{m=0}^{L-1} g_k(n+mN) z^{-m}. \quad (5)$$

Thus, the synthesis filter bank reconstruction can be described by the matrix equation $\mathbf{Z} = \mathbf{Y} \cdot \mathbf{P_s}$, where $\mathbf{Z}$ is the z-transform of the synthesis filter bank output vector shown in Fig. 1. This simple formulation provides a convenient framework in which we can analyze and design efficient filter banks.

A. Modulated Filter Banks

Modulated filters, as stated in Section I, have the form $h_k(n) = h(n) \cdot \Phi(n, k)$, where $h(n)$ is a baseband lowpass filter, $\Phi(n, k)$ is the modulation kernel, and $0 \leq n \leq LN$, $0 \leq k \leq N-1$. There are some natural efficiencies in $h_k(n)$ that are due to the form of the baseband modulation. The approach here, however, seeks a high level of implementation speed by employing the most efficient fast transform algorithms in the filter bank realization. Instead of treating $h_k(n)$ as the product shown above, we consider decomposing it into the form $h_k(n) = h(n) \cdot c(n) \cdot t(m_n, k)$, where $t(n, k)$ is simply a fast block-N transform. More precisely, we require $\Phi(n, k)$ to have the form

$$\Phi(n, k) = c(n) \cdot t(m_n, k) \quad \text{with} \quad 0 \le m_n \le N-1 \tag{6}$$

where m_n is an index mapping function for n in the range of $0, \cdots, N-1$, and $c(n)$ is simply a sequence of real or complex numbers. This decomposition allows the filter bank to be characterized by a fast transform and sparse filter matrices. Given that this condition is true for the analysis and the synthesis filter banks, the filter banks can be expressed as analysis and synthesis polyphase filter matrices $\mathbf{P_a} = \mathbf{F_a} \cdot \mathbf{T_a}$ and $\mathbf{P_s} = \mathbf{T_s} \cdot \mathbf{F_s}$, where $\mathbf{F_a}$ and $\mathbf{F_s}$ are sparse filter matrices containing the coefficients of the analysis and synthesis baseband filters and include $c(n)$. $\mathbf{T_a}$ and $\mathbf{T_s}$ are square transform matrices with elements $t(n, k)$. Once a $t(n, k)$ is determined that satisfies (6), the filter matrices can be obtained trivially from

$$\mathbf{F_a} = \mathbf{P_a} \cdot \mathbf{T_a}^{-1} \tag{7}$$

and

$$\mathbf{F_s} = \mathbf{T_s}^{-1} \cdot \mathbf{P_s}. \tag{8}$$

The key is to determine a $t(n, k)$ that has a fast transform implementation and that also satisfies (6). Φ is best viewed as a rectangular modulation matrix that can be represented by a square transform matrix with the mapping m_n. With a little care, (7) and (8) can be used to obtain good transform matrices $\mathbf{T_a}$ and $\mathbf{T_s}$ by finding $\mathbf{T_a}$ and $\mathbf{T_s}$ such that the resulting filter matrices ($\mathbf{F_a}$ and $\mathbf{F_s}$) are sparse. This paper does not attempt to determine the decomposition for all conceivable $\Phi(n, k)$'s or provide an algorithmic procedure for finding them. Rather, it presents decompositions for several useful cosine kernels, which are derived by careful manipulation of the symmetries and periodicities of the cosine kernel. With these decompositions, a broad class of efficient modulated filter banks emerge, all of which can be designed easily within the framework presented here. As other useful decompositions are discovered, the same framework can be used for design.

B. Perfect Reconstruction

Perfect reconstruction can be achieved by making $\mathbf{P_a}$ and $\mathbf{P_s}$ inverses of each other, which is evident from Fig. 1. For a given analysis polyphase matrix $\mathbf{P_a}$, the synthesis polyphase matrix $\mathbf{P_s}$ is $\mathbf{P_s} = z^{-d} \cdot \mathbf{P_a}^{-1}$, where z^d represents a delay of d blocks or $d \cdot N$ samples. Because the polyphase matrix elements of $\mathbf{P_a}$ are polynomials in z^{-1}, in general, the filter bank resulting from $\mathbf{P_a}^{-1}$ will be noncausal, i.e., the elements will be polynomials in z (with positive exponents). To make them causal, a multiplication by z^{-d} is necessary. Taking the inverse explicitly, we obtain $\mathbf{P_s} = z^{-d} \cdot \mathbf{P_a}^{-1} = \mathbf{T_a}^{-1} \cdot z^{-d} \cdot \mathbf{F_a}^{-1}$. This form can be used to implement the synthesis filter bank. Moreover, if $\mathbf{T_s} = \mathbf{T_a}^{-1}$, then the synthesis filter matrix can be determined directly from $\mathbf{F_a}$, $\mathbf{F_s} = z^{-d} \cdot \mathbf{F_a}^{-1}$. This does imply a structure on the sparse matrix $\mathbf{F_s}$. The impulse responses of the synthesis filters $g_k(n)$ can be determined directly from $\mathbf{P_s}$ because $\mathbf{P_s}$ has elements $P'_{n,k}(z)$, which give $g_k(n)$. In the following subsection, the matrix formulation is illustrated with a simple example.

C. Simple Example

Consider the N-band (N even), $2N$-length, cosine-modulated filter bank with analysis filter vectors $h_k(n) = h(n) \cos\{(\pi/N)(k+0.5)[n+0.5-(N/2)]\} = h(n) \cdot \Phi(n, k)$, where $0 \le n \le 2N-1$. The output of the analysis filter bank is

$$y_k(n) = \sum_{m=0}^{2N-1} x(nN+m)h(m) \cdot \cos\left[\frac{\pi}{N}(k+0.5)\left(m+0.5-\frac{N}{2}\right)\right] \tag{9}$$

which is essentially the TDAC filter bank proposed in [7]. Note that the baseband filter vector $h(n)$ is time reversed compared with the impulse response due to the convolution and that the modulation kernel in this filter bank is closest to the DCT IV [9]. Thus, we choose the transform matrix $\mathbf{T_a}$ to be $t_a(n, k) = \cos[(\pi/N)(k+0.5)(n+0.5)]$, which is the DCT IV transform.

To test if $\mathbf{T_a}$ is a suitable transform, i.e., if it satisfies the condition in (6), the filter matrix $\mathbf{F_a}$ can be computed with (7): From (7), the filter matrix $\mathbf{F_a}$ is given in (9a), which appears at the bottom of the page. This filter matrix is sparse; therefore, $\mathbf{T_a}$ is a suitable transform. Note that here, $c(n)$ is simply a sequence of 1's and -1's. To simplify the computation of the inverse for the synthesis filter bank, we can further decompose $\mathbf{F_a}$ into the delay matrix $\mathbf{D}$ and the coefficient matrix $\mathbf{F}$, where we have (10) and (11), which appear at the bottom of the next page. This results in the analysis equation $\mathbf{Y} = \mathbf{X} \cdot \mathbf{F} \cdot \mathbf{D} \cdot \mathbf{T_a}$.

$$\mathbf{F_a} = \mathbf{P_a} \cdot \mathbf{T_a}^{-1} = \begin{bmatrix} 0 & & h(0)z^{-1} & h(N) & & 0 \\ & \iddots & & & \ddots & \\ h\left(\frac{N}{2}-1\right)z^{-1} & & 0 & & & h\left(N+\frac{N}{2}-1\right) \\ h\left(\frac{N}{2}\right)z^{-1} & & & & & -h\left(N+\frac{N}{2}\right) \\ & \ddots & & & \iddots & \\ 0 & & h(N-1)z^{-1} & -h(2N-1) & & 0 \end{bmatrix}. \tag{9a}$$

The synthesis filter bank consists of the inverse operations, i.e., $\mathbf{T_s} = \mathbf{T_a}^{-1}$ and

$$\begin{aligned}\mathbf{F_s} &= z^{-d} \cdot \mathbf{F_a}^{-1} \\ &= z^{-d} \cdot \mathbf{D}^{-1} \cdot \mathbf{F}^{-1}.\end{aligned} \tag{12}$$

For this simple filter bank, the inverses are very easy to compute: $\mathbf{T_a}^{-1} = \mathbf{T_a} \cdot (2/N)$, which is the inverse DCT, and the inverse of the delay matrix $\mathbf{D}$ is

$$\mathbf{D}^{-1} \cdot z^{-1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & z^{-1} & \\ & & & & \ddots \\ & & & & & z^{-1} \end{bmatrix}. \tag{13}$$

We hasten to point out that the matrix should not contain terms in z, as these are advances, which in turn lead to a noncausal system. Hence, the inverse is multiplied by the delay z^{-1} so that all terms are causal. If the baseband filter $h(n)$ has the symmetries required for the TDAC filter bank (which are not necessary here), then $\mathbf{F}^{-1}$ simplifies to $\mathbf{F}^{-1} = \mathbf{F}^T$, where the superscript "T" denotes the transpose matrix. The resulting synthesis filter matrix is shown in (14), which appears on the next page. This formulation can now be used for an efficient implementation. The synthesis filter coefficients can be obtained from $\mathbf{P_s} = \mathbf{T_s} \cdot \mathbf{F_s}$. Alternatively, the synthesis baseband filter coefficients can be obtained by comparing (14) with $\mathbf{F_s} = \mathbf{T_s}^{-1} \cdot \mathbf{P_s}$. Following the rule we will develop in Section III, the synthesis filters have the form $g_k(n) = h'(n) \cdot (2/N) \cdot \cos\{(\pi/N)(k+0.5)[n+0.5-(N/2)]\}$, where the factor $2/N$ comes from the inverse of $\mathbf{T_a}$. Thus, we have (15), which appears on the next page. Observe that it has the same form as (14). Comparing (14) and (15), it is seen that $h'(n) = h(n)$.

The real issue to address is how to perform the decomposition into the various matrices. The discussion in this subsection illustrated the procedure for the simple case of the N-band $2N$-length modulated filter bank. In the next section, we show the decompositions for several important classes of cosine-modulated analysis/synthesis filter banks with arbitrary lengths. Long filters are represented by $\mathbf{F_a}$ and $\mathbf{F_s}$ matrices, where the coefficients are now polynomials in z^{-1}. Their inverses can be computed analytically as we will show.

III. Extending the Matrix Framework

In this section, some analysis/synthesis filter matrices are presented for several useful cosine modulation functions where the resulting systems can have arbitrary lengths. In particular, we treat filter matrices $\mathbf{F_a}$ and $\mathbf{F_s}$ for several DCT type IV and type II analysis/synthesis systems. The first three cases are based on the DCT type-IV transform matrix. $\mathbf{T_a}$ has the form $t_a(n, k) = \cos[(\pi/N)(k+0.5)(n+0.5)]$ and $\mathbf{T_s} = \mathbf{T_a}^{-1}$.

Case 1: For the first case consider modulated filter banks with filter vectors of the form $h_k(n) = h(n) \cdot \cos\{(\pi/N)(k+0.5)[n+0.5-(N/2)]\}$ and $g_k(n) = h'(n) \cdot (2/N) \cdot \cos\{(\pi/N)(k+0.5)[n+0.5-(N/2)]\}$ with filter lengths $2LN$, where L is an arbitrary positive integer. These filters can be arbitrarily long. From (7), a filter matrix $\mathbf{F_a}$ for longer baseband filters with lengths $(2LN)$ can be computed. The matrix elements have a diamond pattern, and thus, henceforth, such matrices will be called *diamond* matrices for convenience. Specifically, the filter matrix $\mathbf{F_a}$ is given by (15a), which appears on the next page, where

$$P_k(z) = \sum_{m=0}^{L-1} h(m2N+k)(-1)^m z^{-2(L-1-m)}. \tag{16}$$

For the synthesis filter bank, we apply (8) resulting in (16a), which appears on page 1946, with

$$P'_k(z) = \sum_{m=0}^{\infty} h'(m2N+k)(-1)^m z^{-2m}. \tag{17}$$

The indexing of the sum goes to infinity to accommodate the synthesis filters that are now IIR.

The connection between analysis and synthesis for perfect reconstruction is made with the general inverse for the diamond matrix. It can be expressed analytically and is again a

$$\mathbf{D} = \begin{bmatrix} z^{-1} & & & & & \\ & \ddots & & & & \\ & & z^{-1} & & & 0 \\ & 0 & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} \tag{10}$$

and

$$\mathbf{F} = \begin{bmatrix} 0 & & h(0) & h(N) & & 0 \\ & \iddots & & & \ddots & \\ h\left(\frac{N}{2}-1\right) & & 0 & & & h\left(N+\frac{N}{2}-1\right) \\ h\left(\frac{N}{2}\right) & & & & & -h\left(N+\frac{N}{2}\right) \\ & \ddots & & & \iddots & \\ 0 & & h(N-1) & -h(2N-1) & & 0 \end{bmatrix}. \tag{11}$$

diamond matrix. Consider the diamond matrix

$$\mathbf{F_d} = \begin{bmatrix} & & a_0 & b_0 & & \\ & \iddots & & & \ddots & \\ a_{N/2-1} & & & & & b_{N/2-1} \\ a_{N/2} & & & & & b_{N/2} \\ & \ddots & & & \iddots & \\ & & a_{N-1} & b_{N-1} & & \end{bmatrix}$$

where (a_i) and (b_i) are polynomials in z or ratios of polynomials. The inverse is

$$\mathbf{F_d^{-1}} = \begin{bmatrix} & & a'_{N/2-1} & a'_{N/2} & & \\ & \iddots & & & \ddots & \\ a'_0 & & & & & a'_{N-1} \\ b'_0 & & & & & b'_{N-1} \\ & \ddots & & & \iddots & \\ & & b'_{N/2-1} & b'_{N/2} & & \end{bmatrix} \quad (18)$$

with

$$a'_i = \frac{b_{N-1-i}}{a_i b_{N-1-i} - b_i a_{N-1-i}}$$

$$b'_i = \frac{-a_{N-1-i}}{a_i b_{N-1-i} - b_i a_{N-1-i}}$$

where $i = 0 \cdots N-1$.

It can be seen that the inverse of $\mathbf{F_a}$ has the form of $\mathbf{F_s}$ and vice versa, which means that analysis and synthesis for perfect reconstruction both have modulated filters of the same type.

Case 2: Consider N-band, $2LN$-length filter banks based on filter vectors of the form $h_k(n) = h(n)\cos[(\pi/N)(k+0.5)(n+0.5)]$ and $g_k(n) = h'(n)\cdot(2/N)\cdot\cos[(\pi/N)(k+0.5)(n+0.5)]$. From (7) the elements of the analysis filter matrix are nonzero only on the principal diagonal and antidiagonal. Such a matrix structure will be called *bidiagonal* and has the form shown in (18a), which appears at the bottom of the next page, with $P_k(z)$ as defined in (16). The synthesis filter matrix, obtained from (8), also has the same appearance, which is shown in (18b) at the bottom of the next page, with $P'_k(z)$ as defined in (17).

As in the previous case, the connection between analysis and synthesis for perfect reconstruction is made with the general inverse, which again is expressible analytically and

$$\mathbf{F_s} = \begin{bmatrix} & & h\left(\frac{N}{2}-1\right) & h\left(\frac{N}{2}\right) & & \\ & \iddots & & & \ddots & \\ h(0) & & & & & h(N-1) \\ h(N)z^{-1} & & & & & -h(2N-1)z^{-1} \\ & \ddots & & & \iddots & \\ & & h\left(N+\frac{N}{2}-1\right)z^{-1} & -h\left(N+\frac{N}{2}\right)z^{-1} & & \end{bmatrix}. \quad (14)$$

$$\mathbf{F_s} = \mathbf{T_s^{-1}}\cdot\mathbf{P_s} = \begin{bmatrix} & & h'\left(\frac{N}{2}-1\right) & h'\left(\frac{N}{2}\right) & & \\ & \iddots & & & \ddots & \\ h'(0) & & & & & h'(N-1) \\ h'(N)z^{-1} & & & & & -h'(2N-1)z^{-1} \\ & \ddots & & & \iddots & \\ & & h'\left(N+\frac{N}{2}-1\right)z^{-1} & -h'\left(N+\frac{N}{2}\right)z^{-1} & & \end{bmatrix} \quad (15)$$

$$\mathbf{F_a} = \begin{bmatrix} & & P_0(z)z^{-1} & P_N(z) & & \\ & \iddots & & & \ddots & \\ P_{N/2-1}(z)z^{-1} & & & & & P_{N+N/2-1}(z) \\ P_{N/2}(z)z^{-1} & & & & & -P_{N+N/2}(z) \\ & \ddots & & & \iddots & \\ & & P_{N-1}(z)z^{-1} & -P_{2N-1}(z) & & \end{bmatrix} \quad (15a)$$

is a bidiagonal matrix. Consider the bidiagonal matrix

$$\mathbf{F_c} = \begin{bmatrix} a_0 & & & & & b_0 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & a_{N/2-1} & b_{N/2-1} & & \\ & & b_{N/2} & a_{N/2} & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ b_{N-1} & & & & & a_{N-1} \end{bmatrix}.$$

The inverse is

$$\mathbf{F_c^{-1}} = \begin{bmatrix} a'_0 & & & & & b'_0 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & a'_{N/2-1} & b'_{N/2-1} & & \\ & & b'_{N/2} & a'_{N/2} & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ b'_{N-1} & & & & & a'_{N-1} \end{bmatrix} \qquad (19)$$

where

$$a'_i = \frac{a_{N-1-i}}{a_i a_{N-1-i} - b_i b_{N-1-i}}$$

$$b'_i = \frac{-b_i}{a_i a_{N-1-i} - b_i b_{N-1-i}}$$

and $i = 0, \cdots, N-1$.

Here, it can also be seen that the inverse of $\mathbf{F_a}$ has the form of $\mathbf{F_s}$ and vice versa so that analysis and synthesis both have modulated filters of the same type.

Case 3: Consider the filters

$$h_k(n) = h(n) \cdot \cos\left[\frac{\pi}{N}(k+0.5)(n+0.5-N)\right]$$

$$g_k(n) = h'(n) \cdot \frac{2}{N} \cdot \cos\left[\frac{\pi}{N}(k+0.5)(n+0.5-N)\right].$$

Using (7), we obtain the analysis filter matrix shown in (19a), which appears on the next page, with $P_k(z)$ as defined in (16). Following the same approach for the synthesis filter matrix, we obtain (19b), which appears at the bottom of the next page,with $P'_k(z)$ as defined in (17). The connection between $\mathbf{F_a}$ and $\mathbf{F_s}$ for perfect reconstruction is again provided by the general inverse of (19).

In the last two cases, we consider modulated filter banks with a DCT type II, where $\mathbf{T_a}$ has elements $t(n, k) = \cos[(\pi/N)k(n+0.5)]$ and $\mathbf{T_s} = \mathbf{T_a^{-1}}$

Case 4: The fourth case is that of a DCT type-II modulated filter bank with a time shift of $N/2$ in the modulating function, with filters of the form $h_k(n) = h(n)\cos\{(\pi/N)k[n+0.5-(N/2)]\}$ for $k = 0, \cdots, N-1$ and $g_k(n) = h'(n) \cdot (2/N) \cdot \cos\{(\pi/N)k[n+0.5-(N/2)]\}$ for $k = 1, \cdots, N-1$. When $k = 0$, $g_k(n) = h'(n) \cdot (1/N)$. The analysis filter matrix can be written as a diamond matrix, shown in (19c), which appears on the next page, with

$$P_k(z) = \sum_{m=0}^{L-1} h(m2N+k)z^{-2(L-1-m)}. \qquad (20)$$

$$\mathbf{F_s} = \begin{bmatrix} & & P'_{N/2-1}(z) & P'_{N/2}(z) & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ P'_0(z) & & & & & P'_{N-1}(z) \\ P'_N(z)z^{-1} & & & & & -P'_{2N-1}(z)z^{-1} \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & P'_{N+N/2-1}(z)z^{-1} & -P'_{N+N/2}(z)z^{-1} & & \end{bmatrix} \qquad (16a)$$

$$\mathbf{F_a} = \begin{bmatrix} P_0(z)z^{-1} & & & & & -P_N(z) \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & P_{N/2-1}(z)z^{-1} & -P_{N+N/2-1}(z) & & \\ & & -P_{N+N/2}(z) & P_{N/2}(z)z^{-1} & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ -P_{2N-1}(z) & & & & & P_{N-1}(z)z^{-1} \end{bmatrix} \qquad (18a)$$

$$\mathbf{F_s} = \begin{bmatrix} P'_0(z) & & & & & -P'_{2N-1}(z)z^{-1} \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & P'_{N/2-1}(z) & -P'_{N+N/2}(z)z^{-1} & & \\ & & -P'_{N+N/2-1}(z)z^{-1} & P'_{N/2}(z) & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ -P'_N(z)z^{-1} & & & & & P'_{N-1}(z) \end{bmatrix} \qquad (18b)$$

This is reminiscent of Case 1, but with different signs. The synthesis filter matrix is given in (20a), which appears at the bottom of this page, with

$$P'_k(z) = \sum_{m=0}^{\infty} h'(m2N + k)z^{-2m}. \tag{21}$$

The general inverse is as in (18).

Case 5: The last case considered here is that of the DCT type-II with no time shift in the modulating function. Its filter vectors have the form $h_k(n) = h(n)\cos[(\pi/N)k(n+0.5)]$ for $k = 0, \cdots, N-1$ and $g_k(n) = h'(n)\cdot(2/N)\cdot\cos[(\pi/N)k(n+0.5)]$ for $k = 1, \cdots, N-1$. Again, for $k = 0$, $g_k(n) = h'(n)\cdot(1/N)$. It has a bidiagonal analysis filter matrix shown in (21a), which appears at the bottom of the next page, with $P_k(z)$ as defined in (20). The corresponding synthesis filter matrix is given in (21b), which appears on the bottom of the next page, with P'_k as in (21) and the general inverse as in (19).

At this point, we have a matrix structure that describes the analysis and synthesis filter banks for DCT IV and II modulated systems with arbitrary numbers of bands and arbitrary filter lengths. Although the lengths are shown as $2NL$ explicitly, lengths that are not integer multiples of $2N$ are obtainable by restricting the end coefficients of the baseband filter impulse response to be zero. The free parameters are the analysis and synthesis baseband filter coefficients, $h(n)$ and $h'(n)$. These coefficients must be determined such that exact reconstruction is guaranteed. This can be accomplished by taking the inverse of the matrix components as was done in Section II (and illustrated by (12)).

$$\mathbf{F_a} = \begin{bmatrix} P_N(z) & & & & & P_0(z)z^{-1} \\ & \ddots & & & \iddots & \\ & & P_{N+N/2-1}(z) & P_{N/2-1}(z)z^{-1} & & \\ & & P_{N/2}(z)z^{-1} & P_{N+N/2}(z) & & \\ & \iddots & & & \ddots & \\ P_{N-1}(z)z^{-1} & & & & & P_{2N-1}(z) \end{bmatrix} \tag{19a}$$

$$\mathbf{F_s} = \begin{bmatrix} P'_N(z)z^{-1} & & & & & P'_{N-1}(z) \\ & \ddots & & & \iddots & \\ & & P'_{N+N/2-1}(z)z^{-1} & P'_{N/2}(z) & & \\ & & P'_{N/2-1}(z) & P'_{N+N/2}(z)z^{-1} & & \\ & \iddots & & & \ddots & \\ P'_0(z) & & & & & P'_{2N-1}(z)z^{-1} \end{bmatrix} \tag{19b}$$

$$\mathbf{F_a} = \begin{bmatrix} & & P_0(z)z^{-1} & P_N(z) & & \\ & \iddots & & & \ddots & \\ P_{N/2-1}(z)z^{-1} & & & & & P_{N+N/2-1}(z) \\ P_{N/2}(z)z^{-1} & & & & & P_{N+N/2}(z) \\ & \ddots & & & \iddots & \\ & & P_{N-1}(z)z^{-1} & P_{2N-1}(z) & & \end{bmatrix} \tag{19c}$$

$$\mathbf{F_s} = \begin{bmatrix} & & P'_{N/2-1}(z) & P'_{N/2}(z) & & \\ & \iddots & & & \ddots & \\ P'_0(z) & & & & & P'_{N-1}(z) \\ P'_N(z)z^{-1} & & & & & P'_{2N-1}(z)z^{-1} \\ & \ddots & & & \iddots & \\ & & P'_{N+N/2-1}(z)z^{-1} & P'_{N+N/2}(z)z^{-1} & & \end{bmatrix} \tag{20a}$$

With these inverses, it is now possible to construct the synthesis filter bank for any analysis baseband filter and any modulating function that leads to a diamond or bidiagonal filter matrix as long as it is invertible. Conversely, it is possible to construct an analysis filter bank from a given synthesis filter bank given the same conditions. This simple approach of design by taking inverse matrix components is limited, however, because the inverses will generally be IIR and often unstable. Most often, one wishes to have both analysis and synthesis filter banks be FIR. To accommodate this specification, the elements of the filter matrices may be designed iteratively with the constraint that the inverse matrices be FIR. This constraint, as we shall see, is simple to impose and does not represent a practical problem for reasonable filter lengths of interest in real-world applications. In addition, frequency- and time-domain constraints can be imposed easily on the baseband filters to provide desired control over these characteristics. The design framework with its closed-form matrix inverses makes such an iterative design approach fast and effective.

IV. FIR Filter Bank Design

This formulation of the analysis/synthesis problem is constrained structurally to guarantee exact reconstruction. This constraint is valid regardless of the lengths of the filters or the number of bands involved. Thus, we have already addressed several very important aspects of the design flexibility. Now, it is appropriate to discuss the issue of imposing specific time- and frequency-domain characteristics on the filters. Many applications require that the filters in the filter bank have good stopband attenuation, narrow transition width, and perhaps tapered impulse responses or low ripple step responses. Whatever the time- or frequency-domain characteristics are, the baseband filters can be designed by iterative optimization. The procedure involves constructing an error function that represents the frequency- or time-domain error for the analysis and synthesis filter bank and minimizing this error with respect to the filter matrix coefficients. This can be done using a standard library optimization algorithm [4] or by using the specialized optimization algorithm given in the next section.

A. Standard Delay Filter Banks

For analysis filters that are longer than $2N$, the synthesis filters are typically IIR, as pointed out in the last section. In most applications of interest, the preference is generally for FIR synthesis filters. This represents an additional constraint on the design formulation but one that can be handled structurally in the formulation. Observe that if the filter matrix can be written as a product of delay matrices $\mathbf{D}$ of the form shown in (10) and coefficient matrices with real or complex coefficients (as in (11)), the matrix inverses are always FIR. This means both analysis and synthesis filter banks will be FIR. This can be done by using the specific coefficient matrices $\mathbf{F}$ and $\mathbf{C}_i$, which have the form

$$\mathbf{F} = \begin{bmatrix} & & d_0 & d_N & & \\ & \iddots & & & \ddots & \\ d_{N/2-1} & & & & & d_{N+N/2-1} \\ d_{N/2} & & & & & d_{N+N/2} \\ & \ddots & & & \iddots & \\ & & d_{N-1} & d_{2N-1} & & \end{bmatrix}. \tag{22}$$

and

$$\mathbf{C}_i = \begin{bmatrix} c_0^i & & & & & 1 \\ & \ddots & & & \iddots & \\ & & c_{N/2-1}^i & 1 & & \\ & & 1 & c_{N/2}^i & & \\ & \iddots & & & \ddots & \\ 1 & & & & & c_{N-1}^i \end{bmatrix} \tag{23}$$

where $d_0, \cdots, d_{2N-1}$ and $c_0^i, \cdots, c_{N-1}^i$ are real or complex numbers. The 1's on the antidiagonal are a normalization to reduce the number of unknowns without reducing the degree of freedom for the resulting filter bank. The filter design problem can be handled by decomposing the filter matrix $\mathbf{F_a}$ into the cascade of coefficient and delay matrices. $\mathbf{F_a}$ will have the

$$\mathbf{F_a} = \begin{bmatrix} P_0(z)z^{-1} & & & & & P_N(z) \\ & \ddots & & & \iddots & \\ & & P_{N/2-1}(z)z^{-1} & P_{N+N/2-1}(z) & & \\ & & P_{N+N/2}(z) & P_{N/2}(z)z^{-1} & & \\ & \iddots & & & \ddots & \\ P_{2N-1}(z) & & & & & P_{N-1}(z)z^{-1} \end{bmatrix} \tag{21a}$$

$$\mathbf{F_s} = \begin{bmatrix} P'_0(z) & & & & & P'_{2N-1}(z)z^{-1} \\ & \ddots & & & \iddots & \\ & & P'_{N/2-1}(z) & P'_{N+N/2}(z)z^{-1} & & \\ & & P'_{N+N/2-1}(z)z^{-1} & P'_{N/2}(z) & & \\ & \iddots & & & \ddots & \\ P'_N(z)z^{-1} & & & & & P'_{N-1}(z) \end{bmatrix} \tag{21b}$$

form

$$\mathbf{F_a} = \left(\prod_{i=1}^{m} \mathbf{C}_i \cdot \mathbf{D}^2\right) \cdot \mathbf{F} \cdot \mathbf{D}. \tag{24}$$

Such a cascade will result in a diamond filter matrix $\mathbf{F_a}$. The synthesis bank $\mathbf{F_s}$ is computed by simply taking the inverse term by term, resulting in

$$\mathbf{F_s} = \mathbf{D}^{-1} \cdot z^{-1} \cdot \mathbf{F}^{-1} \cdot \left(\prod_{i=0}^{m-1} \mathbf{D}^{-2} \cdot z^{-2} \cdot \mathbf{C}_{m-i}^{-1}\right). \tag{25}$$

The ELT of [14] and [15] is a special case of the above formulation. It results if $\mathbf{F}$ and $\mathbf{C}_i$ are restricted to be orthogonal of one certain type and $\mathbf{T_a}$ and $\mathbf{T_s}$ are a DCT type IV.

B. Controlling the System Delay

Having now treated the issues of exact reconstruction, filter length, number of bands, and baseband filter design, the final aspect remaining is control of the overall system delay. Inspection of (24) and (25) shows that the resulting synthesis filter bank has length $K = 2Nm + 2N$ and the overall system delay is $K-1$, which is the typical system delay for conventional filter banks.

The quintessential factor in analyzing the system delay is the inverse of $\mathbf{D}$. Recall from (13) that the inverse introduces an advance that must be offset with a delay of z^{-1} to keep the system causal. Moreover, since these delays are effectively at the lower sampling rate, each z^{-1} actually corresponds to an N sample delay of the input. This is why the filter length is $K = 2Nm + 2N$.

In the present form, there is no way to vary the system delay without changing the filter lengths. To accommodate the design of systems with other delays, a filter matrix is needed with a form such that its inverse has no positive powers of z. As can be seen from the general inverse for the bidiagonal filter matrix, this can be achieved with the following two matrices: $\mathbf{E}_i$ and $\mathbf{G}_i$, where

$$\mathbf{E}_0 = \begin{bmatrix} 0 & & & & & e_N^0 \\ & \ddots & & & \iddots & \\ & & 0 & e^0_{N+N/2-1} & & \\ & & e^0_{N+N/2} & e^0_{N/2}z^{-1} & & \\ & \iddots & & & \ddots & \\ e^0_{2N-1} & & & & & e^0_{N-1}z^{-1} \end{bmatrix}. \tag{26}$$

The inverse is

$$\mathbf{E}_0^{-1} = \begin{bmatrix} \hat{e}_0^0 z^{-1} & & & & & \hat{e}_N^0 \\ & \ddots & & & \iddots & \\ & & \hat{e}^0_{N/2-1}z^{-1} & \hat{e}^0_{N+N/2-1} & & \\ & & \hat{e}^0_{N+N/2} & 0 & & \\ & \iddots & & & \ddots & \\ \hat{e}^0_{2N-1} & & & & & 0 \end{bmatrix}$$

with

$$\hat{e}_j^0 = \frac{-e^0_{N-1-j}}{e^0_{N+j}e^0_{2N-1-j}}, \quad j = 0, \cdots, N/2-1$$

and

$$\hat{e}^0_{N+j} = \frac{1}{e^0_{2N-1-j}}, \quad j = 0, \cdots, N-1.$$

For $i > 0$, the elements on the antidiagonal are 1 (again a normalization), leading to

$$\mathbf{E}_i = \begin{bmatrix} 0 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 0 & 1 & & \\ & & 1 & e^i_{N/2}z^{-1} & & \\ & \iddots & & & \ddots & \\ 1 & & & & & e^i_{N-1}z^{-1} \end{bmatrix} \tag{27}$$

with inverse

$$\mathbf{E}_i^{-1} = \begin{bmatrix} -e^i_{N-1}z^{-1} & & & & & 1 \\ & \ddots & & & \iddots & \\ & & -e^i_{N/2}z^{-1} & 1 & & \\ & & 1 & 0 & & \\ & \iddots & & & \ddots & \\ 1 & & & & & 0 \end{bmatrix}.$$

The second matrix is

$$\mathbf{G}_i = \begin{bmatrix} g_0^i z^{-1} & & & & & 1 \\ & \ddots & & & \iddots & \\ & & g^i_{N/2-1}z^{-1} & 1 & & \\ & & 1 & 0 & & \\ & \iddots & & & \ddots & \\ 1 & & & & & 0 \end{bmatrix}$$

with inverse

$$\mathbf{G}_i^{-1} = \begin{bmatrix} 0 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 0 & 1 & & \\ & & 1 & -g^i_{N/2-1}z^{-1} & & \\ & \iddots & & & \ddots & \\ 1 & & & & & -g_0^i z^{-1} \end{bmatrix}.$$

A product of the $\mathbf{E}_i$ matrices yields valid filter matrices (i.e., they have the form of a filter matrix) and can support filters with "good" magnitude response characteristics. The analysis filter matrix is of the form

$$\mathbf{F_a} = \prod_{i=0}^{m-1} \mathbf{E}_i. \tag{28}$$

This cascade results in a bidiagonal filter matrix. The synthesis matrix is the inverse

$$\mathbf{F_s} = \prod_{i=0}^{m-1} \mathbf{E}^{-1}_{m-1-i}. \tag{29}$$

The resulting length of the impulse response of the analysis and synthesis filter bank and the delay is

$$K = mN + 0.5N$$

and

$$\text{Delay} = N - 1. \tag{30}$$

The delay that is left is the transform block delay of $N-1$ samples, which is the minimum possible delay.

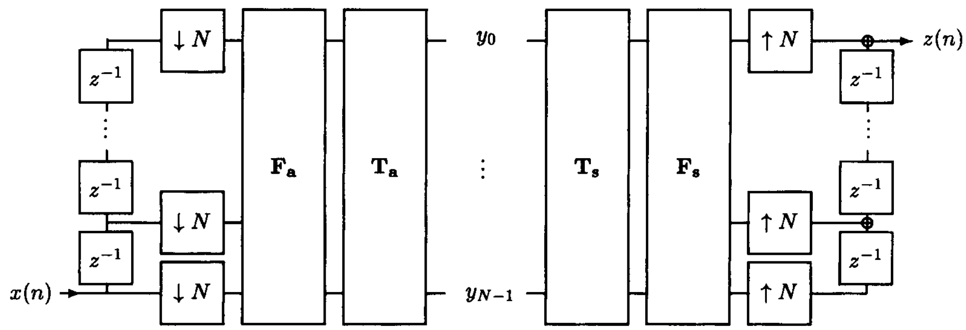

Fig. 2. Modulated filter bank with filter matrices $\mathbf{F_a}$ and $\mathbf{F_s}$.

For filter banks with a higher delay, the desired delay can be achieved by using an appropriate choice of minimum delay and standard delay matrices in cascade. Control of the system delay is now straightforward. One has a choice in using either the minimum delay $\mathbf{G}_i$ or $\mathbf{E}_i$ matrices in conjunction with the normal delay $\mathbf{C}_i$, $\mathbf{D}$, and $\mathbf{F}$ matrices to achieve the targeted system delay. The $\mathbf{G}_i$ matrices should be used here instead of the $\mathbf{E}_i$ matrices because their structure allows analysis-synthesis filters with "good" filter characteristics to be designed. The resulting form is

$$\mathbf{F_a} = \left(\prod_{i=1}^{m} \mathbf{C}_i \cdot \mathbf{D}^2\right) \cdot \mathbf{F} \cdot \mathbf{D} \cdot \left(\prod_{i=1}^{n} \mathbf{G}_i\right) \tag{31}$$

for the analysis filters and

$$\mathbf{F_s} = \left(\prod_{i=0}^{n-1} \mathbf{G}_{n-i}^{-1}\right) \cdot \mathbf{D}^{-1} \cdot z^{-1} \cdot \mathbf{F}^{-1} \cdot \left(\prod_{i=0}^{m-1} \mathbf{D}^{-2} \cdot z^{-2} \cdot \mathbf{C}_{m-i}^{-1}\right) \tag{32}$$

for the synthesis filters. This cascade results in a diamond filter matrix. Given this decomposition, the length and delay of the impulse response are

$$K = m2N + nN + 2N$$

and

$$\text{Delay} = m2N + 2N - 1 \tag{33}$$

samples, respectively. Thus, the independent control over the length and system delay can be specified using the simple formulas in (33). Figs. 2 and 3 show the structure of the filter bank.

C. Analytical Example

The following example illustrates the method. The actual design issues will be discussed in the next chapter. Consider a four-band filter bank ($N = 4$) with a DCT type-IV kernel. The elements of the transform matrix $\mathbf{T_a}$ are $t_a(n, k) = \cos[(\pi/4)(n + 0.5)(k + 0.5)]$ for $n, k = 0, \cdots, 3$ and $\mathbf{T_s} = \mathbf{T_a}^{-1} = \frac{1}{2}\mathbf{T_a}$. Specifically, consider a DCT type-IV filter bank (as in Cases 2 and 3) with length 6 and delay 3. This is a low-delay example. The analysis filter matrix for low delay is $\mathbf{F_a} = \mathbf{E}_0$. The corresponding general filter matrix for a DCT type-IV filter bank is that of Case 2, with no time shift in the modulating function. For a baseband filter length $2N$, it is

$$\mathbf{F_a} = \begin{bmatrix} h(0)z^{-1} & 0 & 0 & -h(4) \\ 0 & h(1)z^{-1} & -h(5) & 0 \\ 0 & -h(6) & h(2)z^{-1} & \\ -h(7) & 0 & 0 & h(3)z^{-1} \end{bmatrix}.$$

For convenience of illustration, the analysis baseband filter impulse response is chosen to have integer coefficients (1, 2, 3, 3, 2, 1, 0, 0). The zeros are appended to pad the filter to length $2NL$ or 8. As seen in (1), the analysis filter vector has the reverse order $h = (0, 0, 1, 2, 3, 3, 2, 1)$, and the analysis filter matrix $\mathbf{F_a}$ becomes

$$\mathbf{F_a} = \mathbf{E}_0 = \begin{bmatrix} 0 & 0 & 0 & -3 \\ 0 & 0 & -3 & 0 \\ 0 & -2 & z^{-1} & 0 \\ -1 & 0 & 0 & 2z^{-1} \end{bmatrix}.$$

The analysis equation is now $\mathbf{Y} = \mathbf{X} \cdot \mathbf{E}_0 \cdot \mathbf{T_a}$, or in the direct form $y_k(n) = \sum_{m=0}^{7} x(4n + m)h(m) \cos[(\pi/4)(k + 0.5)(m + 0.5)]$. The synthesis equation is $\mathbf{X} = \mathbf{Y} \cdot \mathbf{T_s} \cdot \mathbf{Fs} = \mathbf{Y} \cdot \mathbf{T_a}^{-1} \cdot \mathbf{E}_1^{-0}$, where

$$\mathbf{E}_1^{-0} = \begin{bmatrix} -\frac{2}{3}z^{-1} & 0 & 0 & -1 \\ 0 & -\frac{1}{6}z^{-1} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

The corresponding general synthesis filter matrix for a filter bank with a modulating cosine function with a time shift of $-N$ (Case 3) is

$$\mathbf{F_s} = \begin{bmatrix} h'(4)z^{-1} & 0 & 0 & h'(3) \\ 0 & h'(5)z^{-1} & h'(2) & 0 \\ 0 & h'(1) & h'(6)z^{-1} & 0 \\ h'(0) & 0 & 0 & h'(7)z^{-1} \end{bmatrix}.$$

The synthesis baseband filter coefficients are now determined by comparing them with (34), resulting in $h' = (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{2}, -1, -\frac{2}{3}, -\frac{1}{6}, 0, 0)$. The direct-form synthesis

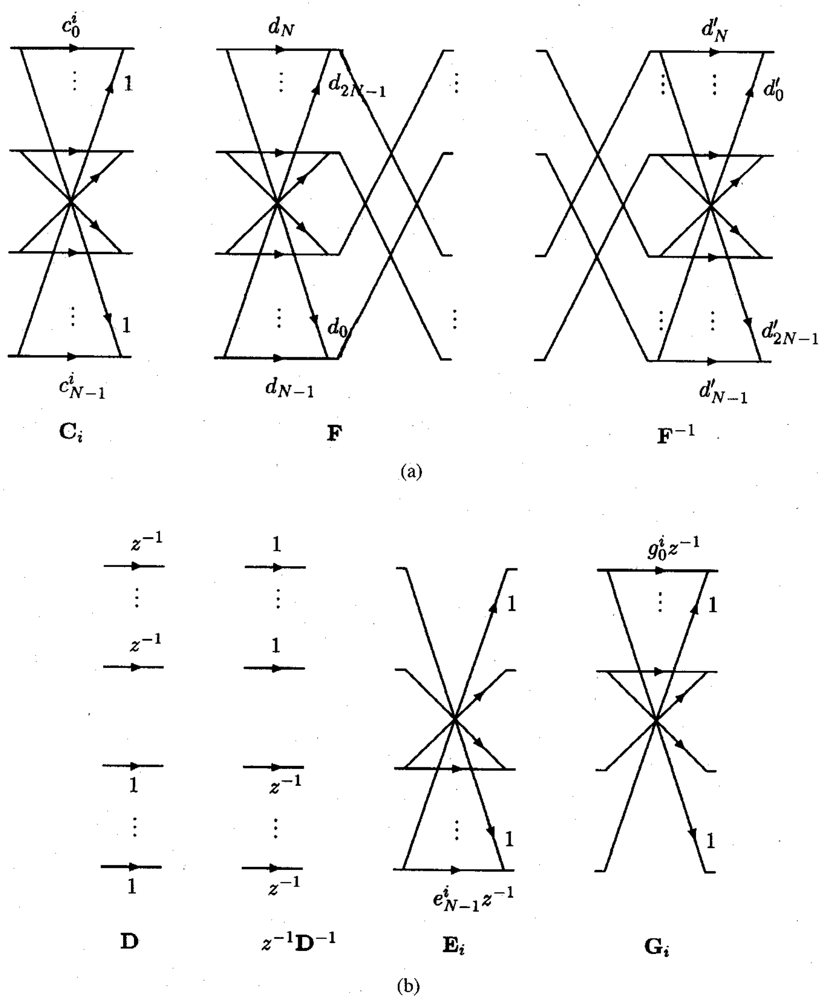

Fig. 3. Structure of some of the elements of the filter matrices.

equation is $x(4n+m) = \sum_{i=0}^{1} h'(4i+m)\frac{1}{2}\sum_{k=0}^{3} y_k(n-i)\cos\left[(\pi/4)(k+0.5)(m+i\cdot 4-4+0.5)\right]$ for $m = 0, \cdots, 3$.

V. Optimization of the Filter Matrix

The objective here is to find the (real) entries of the coefficient matrices for the filter matrix that approximate best some desired frequency or impulse response. A weighted quadratic distance measure was chosen for the error function. In the following, optimization of an error measure based on frequency response characteristics is considered.

The s unknowns of the coefficient matrices are viewed as an s-dimensional row vector $\mathbf{x} = [x_0, \cdots, x_{s-1}]$. The impulse response of length LN of the filter bank resulting from the coefficient matrices with entries from $\mathbf{x}$ are the row vectors $h_{\mathbf{x}}(n)$ for the analysis filter bank and $h'_{\mathbf{x}}(n)$ for the synthesis filter bank, which can be obtained from $\mathbf{P_a}$ and $\mathbf{P_s}$ or $\mathbf{F_a}$ and $\mathbf{F_s}$, respectively. The weighted frequency response at k frequencies ω_i for the analysis filter bank is $H_i = \sum_{n=0}^{LN-1} h_{\mathbf{x}}(n)e^{\omega_i \cdot n}\cdot \mathrm{w}_i$, $i = 0, \cdots, k-1$, where the w_i's are the weights. Similarly for the synthesis bank, $H_i = \sum_{n=0}^{LN-1} h'_{\mathbf{x}}(n)e^{\omega_i \cdot n}\cdot \mathrm{w}_i$, $i = k, \cdots, 2k-1$. The error function is given by

$$f(\mathbf{x}) = \sum_{i=1}^{2k} |H_i(\mathbf{x}) - d_i|^2 = (\mathbf{H}-\mathbf{d})\overline{(\mathbf{H}-\mathbf{d})}^T$$

where d_i is the ideal frequency response at ω_i and represents the concatenation of the analysis and synthesis parts. The overbar denotes complex conjugation. $\mathbf{H}$ and $\mathbf{d}$ are the resulting row vectors for the frequency responses. To optimize the magnitude of the frequency response, the following error function is used:

$$\begin{aligned} f(\mathbf{x}) &= \sum_{i=1}^{2k} (|H_i(\mathbf{x})| - d_i)^2 \\ &= \sum_{i=1}^{2k} \left| H_i(\mathbf{x}) - \frac{H_i(\mathbf{x})}{|H_i(\mathbf{x})|}\cdot d_i \right|^2 \\ &= \sum_{i=1}^{2k} |H_i(\mathbf{x}) - d'_i|^2. \end{aligned}$$

This (nonlinear) function is the one to be minimized. There are several reasonable methods for minimization. The method of conjugate directions (see [25]) was found to have relatively

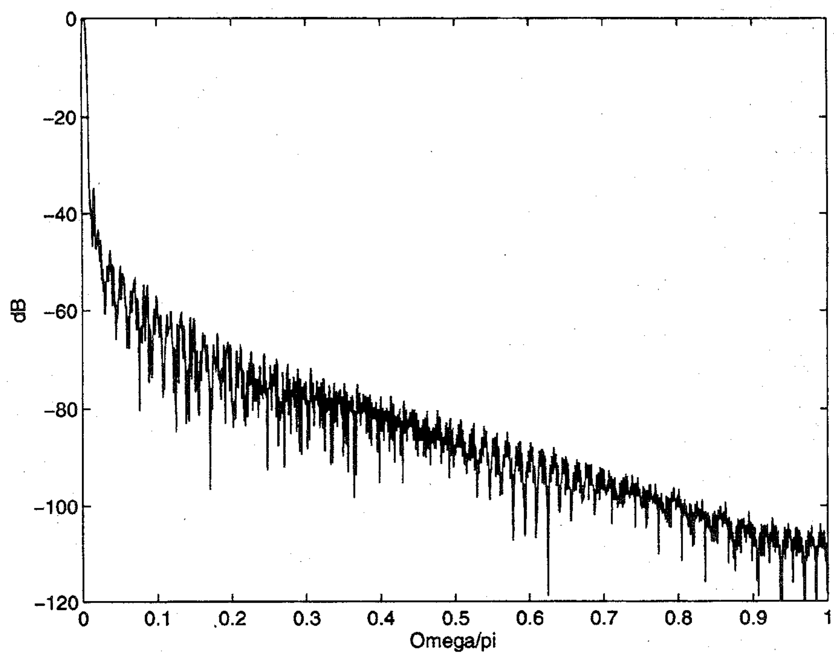

Fig. 4. Frequency response of the analysis and synthesis baseband filter of a low delay filter bank, with a length of 1024 taps, 128 bands, and a system delay of 255 samples.

fast and robust convergence behavior for this function. The method used was a specialized version of it for this quadratic function and is described in the following. It was also used in the design example given in the next section. It consists of a sequence of s independent (1–D) line minimizations. If $\mathbf{v}_i$ is one of the s directions in which f is to be minimized, then one step of Newtons method applied to the first derivative of f can be used to approximate the minimum in that direction. Let $\mathbf{x}_0$ be the starting point of this step. The Newton step is then $\mathbf{x}_1 = \mathbf{x}_0 - \Delta\mathbf{x}$

$$\Delta\mathbf{x} = \frac{\left.\frac{\partial f}{\partial \mathbf{v}_i}\right|_{\mathbf{x}_0}}{\left.\frac{\partial^2 f}{\partial \mathbf{v}_i^2}\right|_{\mathbf{x}_0}} \cdot \mathbf{v}_i.$$

The derivatives can be decomposed as

$$\frac{\partial f}{\partial \mathbf{v}_i} = 2\Re e\left[(\mathbf{H}-\mathbf{d})\,\frac{\overline{\partial \mathbf{H}}^T}{\partial \mathbf{v}_i}\right]$$

and

$$\frac{\partial^2 f}{\partial \mathbf{v}_i^2} \approx 2\Re e\left[\frac{\partial \mathbf{H}}{\partial \mathbf{v}_i} \cdot \frac{\overline{\partial \mathbf{H}}^T}{\partial \mathbf{v}_i}\right].$$

Here, it can be seen that this step approaches a minimum because the second derivative is always greater than zero. If $f(\mathbf{x}_1) > f(\mathbf{x}_0)$, then the magnitude of $\Delta\mathbf{x}$ is reduced, and if that brings no improvement, $\mathbf{x}$ is left unchanged for this $\mathbf{v}_i$.

The directions $\mathbf{v}_i$ are chosen such that a small change in one direction does not change the location of the minimum (i.e., the first derivative) of the other directions. This means that

$$\frac{\partial^2 f}{\partial \mathbf{v}_i \partial \mathbf{v}_j} = 0 \quad \text{for} \quad i \neq j.$$

If $\mathbf{B}$ is the Hessian matrix (or matrix of second derivatives) of f, then $\mathbf{v}_i \mathbf{B} \mathbf{v}_j^T = 0$. This is true for the s eigenvectors of $\mathbf{B}$. $\mathbf{B}$ can be approximated with the first derivative of $\mathbf{H}$. Define $\mathbf{A} = \nabla \mathbf{H}^T$ with its elements as $a_{i,j} = \partial H_j / \partial x_i$. Then, the second derivative (the Hessian matrix) is approximated by neglecting higher order derivatives of $\mathbf{H}$ leading to

$$\mathbf{B} \approx 2\Re e\{\mathbf{A}\overline{\mathbf{A}}^T\}.$$

The complete optimization algorithm can be described by the following pseudo code:

$\mathbf{x}_0$ = row vector of s random numbers
$l = 0$;
Repeat
  $\mathbf{A} = \nabla \mathbf{H}|_{\mathbf{x}_l}^T$
  $\mathbf{v}$ = set of eigenvectors of $\Re e\{\mathbf{A} \cdot \overline{\mathbf{A}}^T\}$
  For $i = 1$ to s
  {
    $\mathbf{dH} = \frac{\partial \mathbf{H}}{\partial \mathbf{v}_i}$;
    $\Delta\mathbf{x} = \Re e\{(\mathbf{H}-\mathbf{d}) \cdot \overline{\mathbf{dH}}^T\} \cdot (\mathbf{dH} \cdot \overline{\mathbf{dH}}^T)^{-1} \cdot \mathbf{v}_i$;
    $j = 0$;
    reduce = false;
    Repeat
      $\mathbf{x}_{l+1} = \mathbf{x}_l - \Delta\mathbf{x}$;
      If $f(\mathbf{x}_{l+1}) \leq f(\mathbf{x}_l)$ then
        reduce = true;
      else $\Delta\mathbf{x} = \Delta\mathbf{x} \cdot 0.1$;
      $j = j + 1$;
    until *reduce* = = *true* or $j > 3$;
    $l = l + 1$;
  }
until $|\mathbf{x}_l - \mathbf{x}_{l-s}|^2 < eps$;
result = $\mathbf{x}_l$;

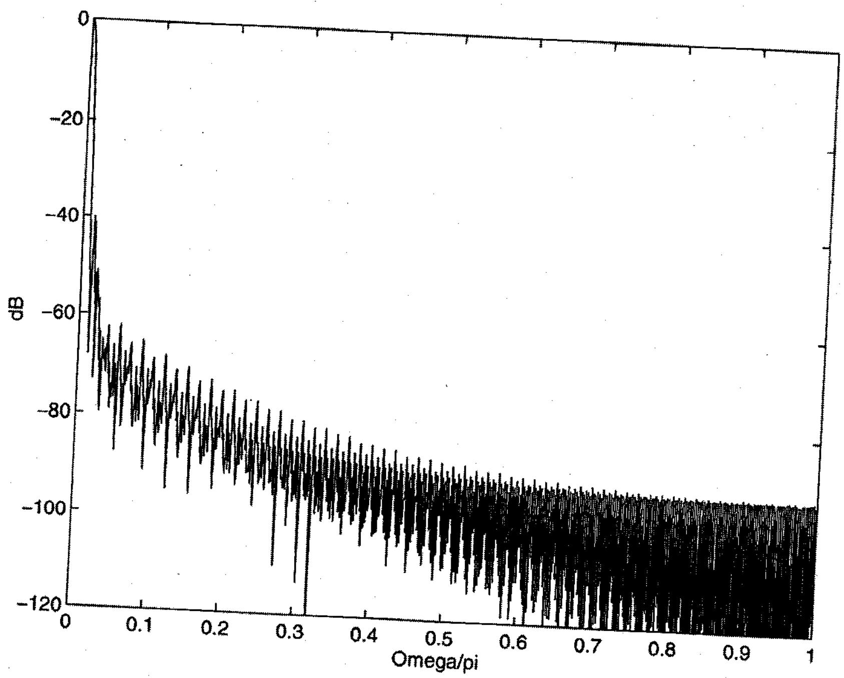

Fig. 5. Frequency response of the analysis and synthesis baseband filter of a filter bank with standard delay, with a length of 768 taps, 128 bands, and a delay of 767 samples.

Note that only the s first derivatives of $\mathbf{H}$ are used. No explicit computations of second derivatives are needed, and no stepsize parameter α is required. This optimization process can be started with a random starting point. To reduce the risk of converging to a poor local minimum, a second random starting point can be tried. The design algorithm usually converges in less than a few hundred iterations and is relatively fast. Experiments have shown that this algorithm is less sensitive to the starting point than others we have tried. However, as with any iterative algorithm, choosing a starting is an issue, especially for designing large filter banks (i.e., many bands, long filters).

The strategy employed in this work that was found to be effective is to start with a smaller filter bank, i.e., to choose m and n to be small, like 0 or 1, and/or with a fraction of the desired numbers of bands, using random initial element values. When the optimization for this smaller filter bank is finished, m or n can be increased in steps of 2, with the coefficients of the added filter matrices initially set to 0, or the number of bands can be increased by increasing the size of the filter matrices, e.g., doubling the size and the number of bands by making pairs of coefficients out of each single coefficient. This is then the starting point for the optimization of the larger filter bank. This process of growing the filter bank can be repeated until the desired size is reached.

VI. Design Examples

In the course of this work, many filter banks were designed by this method. A few of such filters are now highlighted as typical examples. The filter bank whose frequency response is shown in Fig. 4 is a low-delay DCT type-IV system with 128 bands, 1024 taps, and a system delay of 255 samples. The analysis and synthesis filters are of the form $h_k(n) = h(n)\cos[(\pi/N)(k+0.5)(n+0.5-(N/2))]$ and $g_k(n) = h'(n)\cos[(\pi/N)(k+0.5)(n+0.5-(N/2))]$. This corresponds to Case 1 in Section III and is of the form of (31) and (32) with $m = 0$ and $n = 6$. Observe the slope in the stopband attenuation. It was designed for audio coding applications. The magnitude response of the analysis and synthesis filters are identical because the desired frequency response and the weighting function in the optimization were identical.

The next filter bank in Fig. 5 was designed with the same desired frequency response and weighting function but for a standard delay filter bank. It corresponds to the form of (31) and (32) with $m = 2$ and $n = 0$ and has a comparable frequency response at a filter length of 768 taps, but the system delay is now 767 samples instead of the 255 of the previous low-delay example.

As can be seen, filter banks with substantially reduced system delay can be designed without reducing the filter quality. This example was chosen for inclusion because it illustrates the more difficult problem of designing large filter banks, which are used typically in audio coding. Other examples with short lengths may be found in [22]–[24].

VII. Conclusion

A mathematical framework for the treatment of modulated filter banks was introduced in this paper. The framework provides closed-form matrix solutions for reconstruction that can be used in an iterative optimization to design high quality and efficient systems. The formulation is flexible in the sense that there are no fundamental restrictions on the number of bands or the lengths of the filters. Furthermore, the overall system delay is controllable without sacrificing efficiency or flexibility. System delay is controlled by the matrix building blocks using the formula in (30) or (33). By appropriate cascade of these blocks (with their different delays), the target system delay can be achieved. Such a design formulation is expected to be useful in a variety of speech, audio, and image processing applications involving filter banks.

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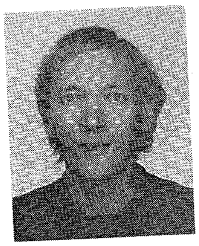

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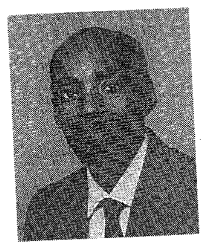

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